AMAT 415 – Tutorial 6

March 5, 2013

This week we take a break from the MATLAB. In this exercise, we're going to calculate

$$I := \int_{-\infty}^{\infty} \frac{dt}{1+t^4}.$$

Try as you might, you won't be able to find an antiderivative of the integrand in terms of commonly used functions. Thus, our method of evaluation will be somewhat indirect. It will involve complex line integrals! This is a prototype for a technique of calculation that we will develop over the next week or two. Except for sketching a few pictures to see what's going on, there not much "work" in this activity. The point is to see how the theory we've developed is applied in a nontrivial example. The only number-crunching is a partial fraction decomposition, and that I suggest outsourcing to the computer. I used Maple, for a change, since it's better for symbolic calculations. Feel free to work in groups. You don't have to hand anything in.

For $R > 0$, let

$$I_R = \int_{-R}^{R} \frac{dt}{1+t^4}$$

so that

$$I = \lim_{R\to\infty} I_R.$$

Let γ_R be the upper half of the circle in $\mathbb{C}$ or radius R, traversed counterclockwise. Thus, R is the initial point of γ_R and $-R$ is its terminal point. Let δ_R be the path along the real axis from $-R$ to R. Let ε_R be the loop in $\mathbb{C}$ obtained by first traversing γ_R and then travelling along δ_R. (Draw a picture.) Therefore,

$$\int_{\varepsilon_R} \frac{dz}{1+z^4} = \int_{\gamma_R} \frac{dz}{1+z^4} + \int_{\delta_R} \frac{dz}{1+z^4}.$$

Using the obvious parametrization of δ_R: $\delta_R(t) = t$ for $t \in [-R, R]$, we see that

$$\int_{\delta_R} \frac{dz}{1+z^4} = \int_{-R}^{R} \frac{dt}{1+t^4} = I_R.$$

Substituting into the previous equation and rearranging, we get

$$I_R = \int_{\varepsilon_R} \frac{dz}{1+z^4} - \int_{\gamma_R} \frac{dz}{1+z^4} =: I_{\varepsilon_R} - I_{\gamma_R}.$$

There is a basic approximation theorem for line integrals that we'll use to bound I_{γ_R}

Theorem 1. *Let γ be a path in $\mathbb{C}$, let $f(z)$ a function that is continuous on γ, and let M be the maximum value of $f(z)$ on γ. Then*

$$\left| \int_\gamma f(z)dz \right| \leq M \cdot \text{length}(\gamma).$$

If z is on γ_R, then z^4 is on the circle or radius R^4. The quantity $|1 - z^4|$ is the distance from 1 to a point on the circle of radius R^4. This is at least $R^4 - 1$. (Why?) Thus, if z is on γ_R and R is big, then

$$\left| \frac{1}{1+z^4} \right| \leq \frac{1}{R^4 - 1}$$

Since γ_R has length πR, the Theorem implies that

$$|I_{\gamma_R}| \leq \frac{\pi R}{R^4 - 1}.$$

Therefore,

$$\lim_{R\to\infty} I_{\gamma_R} = 0$$

and, consequently,

$$I = \lim_{R\to\infty} I_{\varepsilon_R}.$$

We now turn to I_{ε_R}. We use the following theorem that we've employed in class a few times:

Theorem 2. *Let γ be a positively oriented simple loop and let $f(z)$ be a function on an open set containing γ that is analytic except possibly at a finitely many points $z_1, \ldots, z_n$ that all lie inside γ. Then for sufficiently small r,*

$$\int_\gamma f(z)dz = \int_{|z-z_1|=r} f(z)dz + \cdots + \int_{|z-z_n|=r} f(z)dz.$$

The function $1/(1+z^4)$ is analytic except at the four points where the denominator vanishes:

$$z_1 = \frac{1}{\sqrt{2}}(1+i), \quad z_2 = \frac{1}{\sqrt{2}}(-1+i), \quad z_3 = \frac{1}{\sqrt{2}}(-1-i), \quad z_4 = \frac{1}{\sqrt{2}}(1-i).$$

Plot these on the in the complex plane. Note that only z_1 and z_2 are inside the upper half plane, and they lie inside the loop ε_R when $R > 1$. For such R and sufficiently small r,

$$I_{\varepsilon_R} = \int_{|z-z_1|=r} \frac{dz}{1+z^4} + \int_{|z-z_2|=r} \frac{dz}{1+z^4}$$

by the theorem. To evaluate the integrals on the right-hand-side, do a partial fraction decomposition. That is, write

$$\frac{1}{1+z^4} = \frac{A}{z-z_1} + \frac{B}{z-z_2} + \frac{C}{z-z_3} + \frac{D}{z-z_4}.$$

You might want to use a computer to find this decomposition, i.e., A, B, C, and D. The MAPLE command

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convert(1/(1+x^4),parfrac,x,{sqrt(2),sqrt(-1)})
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worked for me.

By Cauchy's theorem,

$$\int_{|z-z_1|=r} \frac{1}{z-z_k} = 0$$

if $k \neq 1$, and similarly for z_2. By a calculation we've done many times now,

$$\int_{|z-z_1|=r} \frac{1}{z-z_1} = 2\pi i,$$

and similarly for z_2. Therefore,

$$I_{\varepsilon_R} = 2\pi i(A+B).$$

(Note the independence of R.) Finally,

$$I = \lim_{R\to\infty} I_{\varepsilon_R} = 2\pi i(A+B).$$